

On Partitions of a Partially Ordered Set

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Using linear programming, we prove a generalization of Greene and Kleitman's generalization of Dilworth's theorem on the decomposition of a partially ordered set into chains.

1. INTRODUCTION

In [7], Greene and Kleitman prove an interesting extension of Dilworth's theorem on decompositions of partially ordered sets. Let P be a finite partially ordered set (where the notation $a < b$ will imply $a \neq b$). Let t be a non-negative integer, and let $f(t)$ be the largest cardinality of a subset S of P satisfying the condition that no more than t elements of S are contained in a chain of P . For any collection \mathcal{C} of disjoint chains C_1, \dots, C_s of P such that $P = \cup C_i$, let $g(\mathcal{C}, t) = \sum_{i=1}^s \min(t, |C_i|)$. (Here and throughout $|S|$ denotes the cardinality of the set S .) Denote by $g(t)$ the minimum of $g(\mathcal{C}, t)$ over all collections \mathcal{C} of disjoint chains whose union is P . It is obvious that $g(t) \geq f(t)$.

THEOREM 1.1 [7]. *In the above notation, $g(t) = f(t)$ for all integers $t \geq 0$.*

Note that Dilworth's theorem is the case $t = 1$. In proving Theorem 1.1, Greene and Kleitman establish another result interesting in its own right.

THEOREM 1.2. [7]. *For every integer $t \geq 0$, there exists a collection \mathcal{C} of disjoint chains whose union is P such that $g(t) = g(\mathcal{C}, t)$ and $g(t+1) = g(\mathcal{C}, t+1)$.*

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The purposes of this note are twofold. In the first place, the form of Theorem 1.1 suggests that it is a special case of the duality theorem of linear programming; likewise, Theorem 1.2 is redolent of concepts from parametric linear programming. We shall show this is indeed the case, so that ideas from linear programming may be substituted for Greene and Kleitman's ingenious combinatorial arguments.

In the second place, the use of linear programming makes it possible to generalize the Greene-Kleitman theorems in a way which we will explain below.

Before doing so, we first remark that Greene in [6] proves analogs of Theorems 1.1 and 1.2, in which the word chain is replaced by antichain (a subset of P no two elements of which are comparable). A generalization of these analogs, based on switch functions, will be given elsewhere. We also note that [10] contains generalizations of Theorems 1.1 and its analog in a different direction.

First, we introduce a nonnegative integral function defined on all chains \mathcal{C} of P . If C and D are chains with at least one element x , we define (see [8] for a similar idea)

$$(C, x, D) = \{y \mid y \in C, y < x\} \cup \{x\} \cup \{y \mid y \in D, x < y\}.$$

Clearly, (C, x, D) is also a chain of P .

DEFINITION 1.1. A nonnegative integer function $r(C)$ defined on all chains of P is said to be a switch function on P if the following hold:

$$\text{if } C \text{ is a subchain of } D, r(C) \leq r(D), \quad (1.1)$$

and

$$\text{if } x \in \mathcal{C} \cap D, r(C) + r(D) = r(C, x, D) + r(D, x, C). \quad (1.2)$$

Note that if r is a switch function, $r + 1$ is also a switch function. Also, the constant function t is a switch function. Let $f(r)$ be the largest cardinality of a subset Q of P such that, for all chains C ,

$$|Q \cap C| \leq r(C). \quad (1.3)$$

For any collection $\mathcal{C} = \{C_1, \dots, C_s\}$ of disjoint chains whose union is P , define

$$g(C, r) = \sum_{i=1}^s \min(r(C_i), |C_i|), \quad (1.4)$$

and

$$g(r) = \min_{\mathcal{C}} g(C, r). \quad (1.5)$$

The results we shall prove, in view of the remarks following Definition 1.1, contain Theorems 1.1 and 1.2.

THEOREM 1.3. *For every switch function r on P ,*

$$g(r) = f(r).$$

THEOREM 1.4. *For every switch function r on P , there exists a collection \mathcal{C} of disjoint chains whose union is P such that*

$$g(r) = g(C, r) \quad \text{and} \quad g(r + 1) = g(C, r + 1).$$

The idea behind the generalization is based on [8], which was an exploitation of the original Ford and Fulkerson concepts in the max flowmin-cut theorem [3]. And the idea behind the proof goes back to the paper by Dantzig and Fulkerson [1], which provided a framework [2] for a (cumbersome) proof of Dilworth's theorem. It is tempting to try to use Fulkerson's elegant proof [4] of Dilworth's theorem to derive Theorems 1.1 and 1.2, but we have not succeeded.

2. PRELIMINARY LEMMAS

We first derive a canonical form for a switch function r .

LEMMA 2.1. *Let $r(a)$ be a nonnegative integral function defined on the elements of P , and $r(a, b)$ a nonnegative integral function defined on all pairs (a, b) where $a < b$. For any chain $C = \{a_0 < a_1 < \cdots < a_s\}$ in P , define*

$$r(C) = r(a_0) + r(a_1, a_2) + \cdots + r(a_{s-1}, a_s). \quad (2.1)$$

Then (2.1) defines a switch function P . Conversely, every switch function on P arises in this way.

Proof. Let C and D be chains, with $x \in C \cap D$. This means

$$\begin{aligned} C &= \{a_0 < a < \cdots < a_{u-1} < x < a_{u+1} < \cdots < a_s\}, \\ D &= \{b_0 < b_1 < \cdots < b_{r-1} < x < b_{r+1} < \cdots < b_t\}. \end{aligned}$$

Then

$$(C, x, D) = \{a_0 < \cdots < a_{u-1} < x < b_{r+1} < \cdots < b_t\}$$

and

$$(D, x, C) = \{b_0 < \cdots < b_{r-1} < x < a_{u+1} < \cdots < a_s\}.$$

Consequently, by (2.1)

$$\begin{aligned}
 & r(C, x, D) + r(D, x, C) \\
 & \quad r(a_0) + r(a_0, a_1) + \cdots + r(a_{u-1}, x) \\
 & \quad + r(x, b_{r+1}) + \cdots + r(b_{t-1}, b_t) \\
 & \quad + r(b_0) + r(b_0, b_1) + \cdots + r(b_{r-1}, x) \\
 & \quad + r(x, a_{u+1}) + \cdots + r(a_{s-1}, a_s) \\
 & = r(a_0) + \cdots + r(a_{u-1}, x) + r(x, a_{u+1}) + \cdots + r(a_{s-1}, a_s) \\
 & \quad + r(b_0) + \cdots + r(b_{r-1}, x) + r(x, b_{r+1}) + \cdots + r(b_{t-1}, b_t) \\
 & = r(C) + r(D),
 \end{aligned}$$

verifying (1.2). Of course (1.1) is obvious.

Conversely, assume r is a switch function, and $r(a)$ is the value of r on the one-element chain $\{a\}$. Let $r(\{a < b\})$ be the value of r on the two-element chain $\{a < b\}$, and define $r(a, b) = r(\{a < b\}) - r(a)$. By (1.1), $r(a, b)$ is nonnegative, so all we need show is that (2.1) holds for all C , which we shall establish by induction on the number of elements in C . We know it holds if $|C| = 1$ or 2. Suppose we know it true if $|C| = s$. Now consider a chain D such that $|D| = s + 1$. Then

$$D = \{a_0 < a_1 < \cdots < a_{s-1} < a_s\}.$$

Let $C = \{a_0 < a_1 < \cdots < a_{s-1}\}$, $E = \{a_{s-1} < a_s\}$.

By (1.2)

$$r(C, a_{s-1}, E) + r(E, a_{s-1}, C) = r(C) + r(E). \quad (2.2)$$

But $(C, a_{s-1}, E) = D$, and $(E, a_{s-1}, C) = \{a_{s-1}\}$, so (2.2) becomes

$$r(D) + r(a_{s-1}) = r(C) + r(E). \quad (2.3)$$

Therefore,

$$r(D) = r(C) + r(E) - r(a_{s-1}).$$

By the induction hypotheses,

$$\begin{aligned}
 r(D) & = r(a_0) + (a_0, a_1) + \cdots + r(a_{s-2}, a_{s-1}) + r(\{a_{s-1} < a_s\}) - r(a_{s-1}) \\
 & = r(a_0) + r(a_0, a_1) + \cdots + r(a_{s-1}, a_s),
 \end{aligned}$$

which verifies (2.1).

The foregoing lemma will be used in proving Theorem 1.3, the next in proving Theorem 1.4. The lemma will give certain sufficient conditions for the “ t -phenomenon”—i.e., Theorems 1.2 and 1.4 or similar results—to hold.

LEMMA 2.2. Let A be an $m \times n$ matrix of rank m , b an m -vector, c and d n -vectors. Assume $P(A, b) \equiv \{x \mid Ax = b, x \geq 0\}$ is not empty and that, for each real t , $0 \leq t \leq 1$, $\min(c + td, x)$, $x \in P(A, b)$ exists. Further, assume that b, d, c are integral, and that the $(m + 1) \times n$ matrix

$$\begin{bmatrix} A \\ d \end{bmatrix}$$

is totally unimodular.

Then there exists an integral vector x^0 such that

$$(c, x^0) = \min_{x \in P(A, b)} (c, x)$$

and

$$(c + d, x^0) = \min_{x \in P(A, b)} (c + d, x).$$

Proof. What we must show is that, in considering the parametric objective function $(c + td, x)$ on $P(A, b)$ (see [5]) there is a vertex optimal for both $t = 0$ and $t = 1$. This vertex will be integral because A is totally unimodular. Choose a value of t (say $\frac{1}{2}$) between 0 and 1, and let x^0 be the vertex which optimizes $(c + \frac{1}{2}d, x^0)$ on $P(A, b)$. To determine all values of t for which this vertex is optimal, one proceeds as follows. Let B be a basis corresponding to x^0 . Find vectors u and v such that

$$u'B = \tilde{c}', v'B = \tilde{d}', \quad (2.4)$$

where \tilde{c} and \tilde{d} are the respective restrictions of c and d to the columns of the basis. If the columns of A are denoted by A_1, A_2, \dots, A_n , then the set of all t for which x^0 minimizes $(c + td, x)$ on $P(A, b)$ is

$$\{t \mid \forall j, c_j + td_j - (u' + tv')A_j \geq 0\}. \quad (2.5)$$

We know (2.5) is nonempty, since $t = \frac{1}{2}$ satisfies all the inequalities in (2.5). We will be done if we can show that, for each j ,

$$c_j + td_j - (u' + tv')A_j = e_j + tf_j,$$

where $f_j = 0, \pm 1$ and e_j is an integer. Now,

$$e_j = c_j - u'A_j = c_j - \tilde{c}'B^{-1}A_j. \quad (2.6)$$

Since A is totally unimodular, $B^{-1}A_j$ is an integral vector; further, c is an integral vector. Hence from (2.6), e_j is integral. We also have

$$f_j = d_j - v'A_j = d_j - \tilde{d}'B^{-1}A_j. \quad (2.7)$$

If A_j is a column of B , $f_j = 0$. So assume A_j not a column of B . Consider the following matrix with $m + 1$ rows and $m + 2$ columns, and denote its

$$M = \left[\begin{array}{c|cc} 0 & & \\ \vdots & & \\ 0 & B & A_j \\ \hline 1 & \tilde{d}' & d_j \end{array} \right]$$

columns by $M_0, M_1, \dots, M_m, M_{m+1}$. The first $m + 1$ columns are linearly independent, so we may write

$$M_{m+1} = a_0 M_0 + a_1 M_1 + \dots + a_m M_m. \quad (2.8)$$

Clearly, $a_0 = f_j$ from (2.7). Further, f_j is an integer, since B is unimodular, and d is integral. If $f_j = 0$, we are done, so assume otherwise. From (2.8), we may write.

$$M_0 = (a_1/f_j) M_1 - \dots - (a_m/f_j) M_m + (1/f_j) M_{m+1}. \quad (2.9)$$

But the matrix formed by columns M_1, \dots, M_{m+1} is unimodular, by hypothesis. Hence, from (2.9), $1/f_j$ is also an integer. Hence, $f_j = \pm 1$.

We remark that, just as the work pioneered by Fulkerson and Edmonds showed that the uses of linear programming in polyhedral combinatorics need not be confined to cases where the matrix of inequalities was totally unimodular, it seems reasonable to believe that interesting instances of the t -phenomenon can arise in cases where the hypotheses of this lemma are not satisfied.

3. PROOF OF THEOREM 1.3

Our approach is to apply the duality theorem to a suitably chosen transportation problem with $n + 1$ rows and columns, indexed $0, 1, \dots, n$, and where $1, \dots, n$ refer to the n elements of P . The 0 th row and columns have sum n , all other rows and columns have sum 1. The costs c_{ij} are given as follows:

$$\begin{aligned} c_{00} &= c_{1,0} = \dots = c_{n0} = 0, \\ c_{0j} &= r(j), \quad j = 1, \dots, n, \end{aligned}$$

where $r(j)$ comes from Lemma 2.1;

$$c_{ii} = 1, \quad i = 1, \dots, n,$$

$$c_{ij} (i \neq j) = \begin{cases} \infty & \text{if } i \not\leq j, \\ r(i, j) & \text{if } i \leq j \text{ (Lemma 2.1).} \end{cases}$$

In the usual fashion of exhibiting transportation problems in a table listing costs and sums, we have

0	$r(1)$. . .	$r(n)$	n
0	1	$r(1,2) \dots$	$r(1,n)$	1
.	$r(2,1)$.	.	.
.
.	.	.	$r(n-1,n)$.
0	$r(n,1)$. . .	1	1
n	1	. . .	1	

In this table, if $i \not\leq j$, $r(i, j)$ should be replaced by ∞ .

We now minimize $\sum_{i=0}^n \sum_{j=0}^n c_{ij} x_{ij}$, subject to

$$x_{ij} \geq 0, \quad \text{all } i \text{ and } j,$$

$$\sum_j x_{0j} = \sum_j x_{j0} = n,$$

$$\sum_j x_{ij} = 1 \quad \text{for all } i = 1, \dots, n,$$

$$\sum_i x_{ij} = 1 \quad \text{for all } j = 1, \dots, n.$$

At a minimizing vertex, all x_{ij} are integers. Clearly all x_{ij} other than x_{00} will be 0 or 1, and $x_{ij} = 0$ if $i \not\leq j$. Let $x_{0j_1} > 0$ for some $j_1 > 0$. Then $x_{j_1 j_2} = 1$ for some $j_2 \neq j_1$. If $j_2 = 0$, stop. Otherwise, $x_{j_2 j_3} = 1$ for some $j_3 \neq j_1, j_2$. If $j_3 = 0$, stop. Otherwise, continue in this fashion. Eventually, we must stop. Then we have a chain C of P , $C\{j_1 < j_2 < \dots < j_r\}$ (when $x_{j_r 0} = 1$), where $x_{0j_1} = x_{j_1 j_2} = \dots = x_{j_{r-1} j_r} = x_{j_r 0} = 1$, and the contribution of these positive x_{ij} to the objective function is $r(C)$ by Lemma 2.1. (Note that all other entries in rows and columns j_1, \dots, j_r are 0.)

In this manner, from each nonzero x_{0j} , $j > 0$, we construct a corresponding

chain. This gives us a collection of disjoint chains C_1, \dots, C_r of P . Now consider those (i, j) such that $x_{ij} = 1$, $i > 0$, $j > 0$, but neither i nor j is contained in $C_1 \cup C_2 \cup \dots \cup C_r$.

Suppose there is such an $x_{i_1 i_2} = 1$, $i_1 \neq i_2$. Then we must have a cycle $i_1 < i_2 < i_3 < \dots < i_1$, where all $i_k > 0$. But this is impossible, since P is partially ordered. Hence the only nonzero remaining elements x_{ij} , $i > 0$, $j > 0$ are all x_{ii} , $i \notin \bigcup_{i=1}^r C_i$. Let us think of these as one-element chains C_{r+1}, \dots, C_{r+s} . As for x_{00} , whatever its value, it contributes nothing to the objective function since $c_{00} = 0$. Thus the value of the objective function is

$$\sum_{i=1}^r r(C_i) + \sum_{i=r+1}^{r+s} |C_i|. \quad (3.1)$$

If we let $\mathcal{C} = \{C_1, \dots, C_{r+s}\}$, \mathcal{C} is a collection of disjoint chains including all elements of P . We claim (3.1) is $g(\mathcal{C}, r)$.

Suppose, for $1 \leq i \leq r$, $|C_i| < r(C_i)$. Let $C_i = \{a_1 < a_2 < \dots < a_q\}$, which means

$$x_{0a_1} = x_{a_1 a_2} = \dots = x_{a_q 0} = 1.$$

Set these x 's to 0, replace the 0 value of $x_{a_1 a_1}, \dots, x_{a_q a_q}$ by 1; change x_{00} to $x_{00} + 1$, and leave all other x 's unchanged. The row and column sum conditions will still be satisfied, and the value of the objective function decreased.

Similarly, suppose for $r+1 \leq i \leq r+s$, $r(C_i) = r(i) < |C_i| = 1$; i.e., $r(i) = 0$. Then change x_{ii} from 1 to 0, change x_{0i} and x_{i0} from 0 to 1, change x_{00} to $x_{00} - 1$, and the objective function is decreased. (Note that as long as $x_{ii} = 1$ for some i , $x_{00} > 0$.)

Therefore, the value of the objective function is $g(\mathcal{C}, r)$ for some \mathcal{C} .

We shall now prove that the value of the objective function in the dual problem is $|Q|$ for some $Q \subset P$ satisfying (1.3). Since Q satisfying (1.3) and \mathcal{C} a collection of disjoint chains covering P implies $|Q| \leq f(r)$, the duality theorem will show $|Q| = f(r)$, which will prove Theorem 1.3.

The dual problem is

$$\text{maximize } n(\xi_0 + \eta_0) + \sum_{i>0} \xi_i + \sum_{j>0} \eta_j,$$

where $\xi_i + \eta_j \leq c_{ij}$.

Clearly, we may set ξ_0 to 0 without disturbing either the objective function or the inequalities. Since $c_{00} = 0$, $\eta_0 \leq 0$. If $\eta_0 < 0$, replace it by 0, and lower all ξ_i , $i > 0$, by $-\eta_0$. The inequalities are still satisfied and the objective function is unchanged. Hence, we may assume $\xi_0 = \eta_0 = 0$.

We claim that, for each $i = 1, \dots, n$,

$$\xi_i + \eta_i = 0 \text{ or } 1. \quad (3.2)$$

Suppose (3.2) false, i.e., for some i ,

$$\xi_i + \eta_i < 0. \quad (3.3)$$

Since our problem is to maximize, (3.3) would permit us to raise ξ_i unless

$$\xi_i + \eta_j = c_{ij} \quad \text{for some } j \neq i. \quad (3.4)$$

Similarly, we may assume

$$\xi_k + \eta_i = c_{ki} \quad \text{for some } k \neq i. \quad (3.5)$$

Suppose $j > k > 0$. Then (3.4) and (3.5) become

$$\xi_i + \eta_j = r(i, j), \quad \xi_k + \eta_i = r(k, i). \quad (3.6)$$

But $k < i < j$ implies $k < j$, so

$$\xi_k + \eta_j \leq r(k, j). \quad (3.7)$$

Now (3.3), (3.6), and (3.7) imply $r(i, j) + r(k, i) = (\xi_i + \eta_i) + (\xi_k + \eta_j < r(k, j))$.

Therefore, $r(\{k < i < j\}) = r(k) + r(k, i) + r(i, j) < r(\{k < j\}) = r(k) + r(k, j)$, which violates (1.1).

Next, assume $j = 0, k > 0$. Then (3.4) and (3.5) become

$$\xi_i = r(i), \quad \xi_k + \eta_i = r(k, i).$$

Together with (3.3) and $\xi_k + \eta_0 = \xi_k \leq 0$, this means $r(i) + r(k, i) = \xi_i + \eta_i + \xi_k < r(k, i)$, which implies $r(i) < 0$, impossible.

Next, assume $j > 0, k = 0$. From (3.4) and (3.5), $\xi_i + \eta_j = r(i, j)$, $\eta_i = r(i)$. Therefore, $r(i) + r(i, j) < r(j)$, violating (1.1).

Finally, if $j = 0$ and $k = 0$, we have from (3.4) and (3.5) $\xi_i = 0, \eta_i = r(i)$, so $\xi_i + \eta_i = r(i)$, which cannot be negative. So (3.2) is true.

Let $Q = \{i \mid i > 0, \xi_i + \eta_i = 1\}$. We will be done if we prove (1.3) holds for all chains C . Let $C = (\{a_1 < a_2 < \cdots < a_s\})$. Recall $\xi_0 = 0 = \eta_0$. Then

$$\begin{aligned} |Q \cap C| &\leq \xi_0 + \eta_0 + \sum_1^s \xi_{a_i} + \sum_1^s \eta_{a_i} \\ &= (\xi_0 + \eta_{a_1}) + (\xi_{a_1} + \eta_{a_2}) + \cdots + (\xi_{a_{s-1}} + \eta_{a_s}) + (\xi_{a_s} + \eta_0) \\ &\leq r(a_1) + r(a_1, a_2) + \cdots + r(a_{s-1}, a_s) + 0 \\ &= r(C), \end{aligned}$$

which is (1.3).

4. PROOF OF THEOREM 1.4

Retain the same transportation problem as in the preceding section, except that the entries $r(1), \dots, r(n)$ in the 0th cost row are replaced by $r(1) + t, \dots, r(n) + t$. We must show that there is a vertex which minimizes the objective function when $t = 0$ and when $t = 1$. Let $C = \{c_{ij}\}$ be the cost vector of the original problem, $d = \{d_{ij}\}$ be defined by

$$d_{01} = \dots = d_{0n} = 1, \quad \text{all other } d_{ij} \text{ are } 0.$$

We are minimizing $\sum_{i=0}^n \sum_{j=0}^n (c_{ij} + td_{ij})x_{ij}$, where

$$\begin{aligned} x_{ij} &\geq 0, \\ \sum_j x_{ij} &= 1, \quad i = 1, \dots, n, \\ \sum_i x_{ij} &= 1, \quad j = 1, \dots, n, \\ \sum_i x_{i0} &= n. \end{aligned} \tag{4.1}$$

Note that we have not included in (4.1) the equation $\sum_j x_{0j} = n$, since it is implied by the others. Thus the matrix of the Eqs. (4.1) has $2n + 1$ rows and is of rank $2n + 1$. All entries in that matrix A are $(0, 1)$, all data are integral, and the matrix

$$M = \begin{bmatrix} A \\ d \end{bmatrix}$$

is totally unimodular, since the rows of the $(0, 1)$ M can be partitioned into two parts, such that every column has at most two nonzeros, and if two occur they are in different parts [9]. Hence, Lemma 2.2 applies and we are done.

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